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Divisia Indexes, Money and Welfare¹²

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Abstract

We suggest the use of a particular Divisia index for measuring welfare losses due to interest rate wedges and inflation. Compared to the existing options in the literature: i) when the demands for the monetary assets are known, closed-form solutions for the welfare measures can be obtained at a relatively lower algebraic cost; ii) less demanding integrability conditions allow for the recovery of welfare measures from a larger class of demand systems and; iii) when the demand specifications are not known, using an index number entitles the researcher to rank different vectors of opportunity costs directly from market observations. We use two examples to illustrate the method.

1 Introduction

We suggest the use of a particular Divisia index for measuring welfare losses due to interest rate wedges and inflation. Compared to the existing options in the literature: i) when the demands for the monetary assets are known, closed-form solutions for the welfare measures can be obtained at a relatively lower algebraic cost; ii) less demanding integrability conditions allow for the recovery of welfare measures from a larger class of demand systems and; iii) when the demand specifications are not known, using an index number entitles the researcher to rank different vectors of opportunity costs directly from market observations. We use two examples to illustrate the method.

We denote by “deposits” all the monetary assets in the economy other than currency. Deposits can be interest-bearing or non-interest-bearing. Households can acquire monetary services by holding currency or by holding deposits. Compared to the benchmark asset in the economy, which we call bonds, and which does not perform monetary services, currency and deposits, by definition, pay lower interest rates. We define as interest-rate wedge the difference between the highest interest rate in the economy, paid by bonds, and the interest rate paid by a specific monetary aggregate.

High interest-rate wedges between bonds and currency (or deposits) imply lower holdings of monetary assets by households. Since these monetary assets save households shopping time that can be directed to the production of consumption goods, interest-rate wedges lead to a decline of welfare.

We define and suggest the use of a particular Divisia index as an approximation to estimate the welfare costs of interest-rate wedges. Under some particular hypotheses, these wedges can be solely determined by the rate of inflation. In this particular case, on which we concentrate in Section 7 of this work, the results derived here can also be used in what regards the welfare costs of inflation.

We will not focus on how the interest-rate wedges are formed. We want to concentrate on behavior of households that take the vector of opportunity costs (here, a synonym for interest rate wedges) of different monetary assets as given.

An integrated general-equilibrium approach would consider interest rates paid by the different monetary assets in a deterministic economy, like ours, as depending upon variations in the rate of monetary expansion or, say, innovations of the production possibility set. This would imply the necessity of modelling a non-costless banking sector and of considering additional real resources being withdrawn from the production of the consumption good. We do not pursue these paths here.

Except in Section 7.2, where we analyze the case of the welfare costs of

inflation under competitive assumptions, we abstract from a banking system. We assume that the government issues bonds and all monetary liabilities. As Calvo and Végh (1996, pp. 1) put it: “In high inflation countries, policy-makers often end up paying interests on part of the money supply”. In our model, as it also happens in Calvo and Végh (op. cit.), the government is assumed to set all the interest rates paid by the monetary assets (but not of bonds), in a discretionary way. By doing that, the government gives up control of the composition of its liabilities.

The hypothesis that the government pays interests on money is equivalent to assuming that the lump-sum transfers, which are usually used, in general-equilibrium analyses, for the purpose of introducing money, are proportional to the money holdings. This assumption is also sometimes used in the literature to generate superneutrality of some monetary models.

We provide three different welfare measures, all of which can generate closed-form solutions (although at different algebraic costs), and two of which can be approximated by Divisia indexes generated from direct market observations. These measures are closely related to previous results derived by Simonsen and Cysne (1994, 1999).

The first option, given by equation (16), generalizes Lucas’ (2000, pp. 265) equation (5.8), which has been derived for an economy without interest-bearing deposits, to an economy where deposits can pay interests. The generalization implies substituting a system of simultaneous non-separable partial differential equations for Lucas’ ordinary differential equation (5.8).

The second option (equation (9)), which we suggest as a reasonable and easier-to-calculate alternative to (16), is based on an approximation of this equation by a Divisia line integral defined in the n -dimensional Euclidean space. This Divisia measure has two important advantages.

First, when the demand functions for the different assets are known, closed-form solutions can be obtained at a lower algebraic cost. Indeed, solving line integrals is generally much simpler than solving a system of non-separable, non-linear partial differential equations.

Second, when the assets demands are not known, or are costly to obtain, approximating a welfare measure by an index number allows the researcher to rank different vectors of opportunity costs directly from market observations of interest rates and monetary aggregates.

Solely for the purpose of comparison with these two previous alternatives, we also present and analyze a third option, given by the multidimensional consumer’s surplus measure (equation (10)). This measure constitutes an upper bound to the two previous measures above mentioned, and is less precise than (9) in approximating the original measure (16). Therefore, for practical purposes, equation (9) dominates equation (10).

The existence of closed-form solutions for (9) and (10) is guaranteed by the property of path independence of the respective underlying line integrals. In each case, this property implies the existence of a potential function that turns out to be the desired closed-form welfare measure.

Our basic model is drawn from Simonsen and Cysne (1994,1999), which in turn draws on the shopping-time version of Lucas' (2000, Section 5) approach to the welfare costs of inflation. Lucas attributes to McCallum Goodfriend (1987) the basic framework of the shopping-time economy that he uses in his investigations.

We generalize the economy of Simonsen and Cysne (1994,1999) in two different directions. First, where Simonsen and Cysne allow for only two monetary assets, we allow for several. Second, and more important, here the opportunity cost of deposits is allowed to vary, when deriving the basic inequalities (20) and (21). These inequalities are the ones that formally characterize the approximation of the non-separable partial differential equations by a line integral. Allowing the opportunity cost of deposits to change has important technical consequences. Particularly, it implies the substitution, in the calculation of welfare variations, of line integrals in multidimensional spaces for Riemann unidimensional integrals.

Cysne (2000) investigates the integrability of multidimensional consumer's surplus in a similar setting, considering an economy that has currency and one type of deposit. His basic result is that the blockwise weak separability of the transacting technology is a necessary and sufficient condition for a coherent definition of the multidimensional consumer's surplus. Here we assume such a hypothesis regarding the transacting technology.

Our economy is a representative-agent economy, where the following hypotheses are maintained: (1) blockwise weak separability of the monetary aggregates in the transacting technology function; (2) first-degree homogeneity of the monetary aggregator in the transacting-technology function, and (3) absence of uncertainty and of capital.

Empirical evaluations based on the results here derived should take into consideration these simplifying assumptions. Modifying each of these hypothesis constitutes topics for further research in the area. Particularly, the transacting technology assumes a unit income elasticity in the monetary aggregator, a hypothesis that must be tested.

Financial innovations, when present, must also be properly addressed. Given the importance of this issue in empirical evaluations, Subsection 4.1.2 briefly discusses a possible correction for DS when non-neutral technical progress autonomously reduce the demands for monetary assets.

Divisia indexes depend on the normalization (or deflation) of the nominal prices used in their construction (Bruce (1977)). In this work, the interrela-

tions among welfare measures and Divisia indexes are presented with respect to three different versions of such indexes. In the first case, all nominal opportunity costs are normalized by what we define as seigniorage. We denote this version of the Divisia index by DS (S standing for Seigniorage). In the second case, the normalization is performed by what we define as “Extended GDP”, the sum of the potential GDP (which in our model is normalized to one) and the seigniorage. We refer to the Divisia index so defined as DE (E standing for Extended). Thirdly, we normalize the weights of the Divisia index by the potential GDP, in which case we use the denomination DG (G standing for GDP). DE and DS correspond, respectively, to the second and third options of welfare measures that we mentioned before (equations (9) and (10)). DG is presented only for completeness. This measure exactly tracks first-degree-homogenous monetary aggregator functions, up to an arbitrary normalization.

The remainder of this work is organized as follows. Section 2 presents the model. Section 3 is used to define three different versions of the Divisia index of monetary services, and Section 4 to investigate their path independence. Section 4 also concentrates on the issue of financial innovations and on how DS can be used to rank different price vectors. In Section 5 we derive the relations between the Divisia indexes and the welfare costs of interest rate wedges. Section 6 exemplifies the use of the different welfare measures in applied work and briefly discusses the necessity of using discrete-time approximations for DE when the assets demand functions are not known. Section 7 presents alternative hypotheses that allow the results here derived to be used in the investigations of the welfare costs of inflation. Finally, Section 8 offers the conclusions of the work.

2 The Model

- *Households and Firms*

We consider an economy with n different assets performing monetary functions. We also consider bonds, which are used only as a store of value. Bonds pay the benchmark interest rate i . Each other asset is supposed to have a different degree of moneyiness. In equilibria with strictly positive demands, which are the ones that we consider here, opportunity costs, defined relatively to the benchmark interest rate, are proportional to the productivity of each asset in providing monetary services. Therefore, in equilibrium, one can positively associate the opportunity costs of each asset with its degree of moneyiness. In this way, currency and as well as non-interest-bearing

deposits are the assets with the highest opportunity cost and the highest degree of moneyiness. We denote the monetary assets (which correspond to the totality of assets, except for bonds) by the n -dimensional vector $X = (X_1, X_2, \dots, X_n)$, and their real quantities by the vector $x = (x_1, x_2, \dots, x_n)$.

Each asset x_1, x_2, \dots, x_n pays an interest rate i_1, i_2, \dots, i_n . We make $\vec{i} = (i_1, i_2, \dots, i_n)$. Relatively to the benchmark rate, the vector of opportunity costs $u = (u_1, u_2, \dots, u_n)$ is given by, respectively, $i - i_1, i - i_2, \dots, i - i_n$. For the remainder of this text, we assume that $u \in R_{++}^n$ and $x \in \bar{R}_{++}^n$, with each x_i , separately or together, being able to assume values such that its marginal productivity in providing monetary services is equal to zero (possibly $+\infty$). One can think of the first monetary asset x_1 as being composed by currency (x_{1c}) plus non-interest bearing deposits (x_{1d}), in which case $x_1 = x_{1c} + x_{1d}$, the interest rate $i_1 = 0$, and the opportunity costs $i - i_1 = i$.

In this model the potential GDP, which takes place when the opportunity cost vector approaches the zero vector, is normalized to unity. Exogenous growth at a rate λ , like in Lucas (2000), can be added at no cost. Since it does not add to the understanding of the question we want to investigate, we make $\lambda = 0$. We define the inner product $\langle u, x \rangle$ as the seigniorage. From the point of view of the representative consumer, the seigniorage denotes the total opportunity cost, at a point in time, of holding the monetary assets vector x . We also define the Extended GDP as $GDE = Potential\ GDP + \langle u, x \rangle = 1 + Seigniorage$.

The representative household is assumed to maximize:

$$\int_0^\infty e^{-gt} U(c) dt$$

where $U(c)$ is a concave function of the consumption at instant t and $g > 0$. The household is endowed with one unit of time that can be used to transact or to produce the consumption good, so that $y + s = 1$, where y stands for the production of the consumption good and s for the fraction of the initial endowment spent as transacting time. Households can accumulate bonds (B) and n different types of monetary assets. In their utility maximization, households take as given the nominal interest rate on bonds, i , and the opportunity costs of holding monetary assets, $u = (u_1, u_2, \dots, u_n)$. Letting $P = P(t)$ be the price of the consumption good, the household faces the budget constraint:

$$\sum_{i=1}^n \dot{X}_i + \dot{B} = iB + \langle \vec{i}, X \rangle + P(y - c) + H \quad (1)$$

where H indicates the (exogenous) flow of income transferred to the household by the government. Making $\pi = \dot{P}/P$ (inflation rate), $x = (x_1/P, x_2/P, \dots, x_n/P)$, $i_R = (i_1 - \pi, i_2 - \pi, \dots, i_n - \pi)$ and $h = H/P$, the budget constraint reads:

$$\dot{b} + \sum_{i=1}^n \dot{x} = 1 - (c + s) + h + (i - \pi) \quad b + \langle i_R, x \rangle$$

The consumer is also subject to the transacting-technology constraint:

$$c = N(x, s) \quad (2)$$

We assume that $N(x, s)$ is globally blockwise weakly separable with respect to the vector x and the variable s . The formal definition and an encompassing analysis of separability can be found in Leontief (1947). In our case, this means that, for any i and j , $\frac{\partial}{\partial s}(N_{x_i}/N_{x_j}) = 0$. We assume the particular case of separability:

$$c = N(x, s) = G(x)\phi(s). \quad (3)$$

with $\phi(0) = 0$, $\phi'(s) > 0$, $\phi''(s) \leq 0$. We also assume that $G(x)$ is differentiable, first degree homogeneous, and strictly increasing in each of its variables, with decreasing marginal returns. Since, in equilibrium, $c = y = 1 - s$, $(1 - s)/\phi(s) = G(x)$. When any x_i tends to infinity, s tends toward zero, the only point where $\phi(s)$ is allowed to assume a zero value. In the steady state, the necessary conditions for optimization are given by the equilibrium equation (4) and by the first order conditions (5) and (6):

$$1 - s = G(x)\phi(s) \quad (4)$$

$$i = \pi + g \quad (5)$$

$$G_{x_i}(x)\phi(s) = u_i G(x)\phi'(s), \quad i = 1, 2, \dots, n \quad (6)$$

Equation (5) establishes the link between the rate of inflation and the benchmark interest rate. In the steady state, the rate of inflation is determined so that the seigniorage matches the transfers (h) plus net real interest payments made by the government.

We use equations (4) and (6) to determine the $n + 1$ variables $u(x)$ and $s(x)$, in which case the respective Jacobian is a positive definite diagonal matrix. Since we will be interested in ensuring that these functions are globally integrable, we assume that they are defined over an open and connected set $L \subset \bar{R}_{++}^n$, which is also simply connected.

Also, notice that, by using the homogeneity of G and Euler's theorem, one can write:

$$\phi(s) = \langle u, x \rangle \phi'(s) \quad (7)$$

- *Government*

Like in Lucas (2000), we consider this to be an economy with lump sum taxation, where the government can implement any given interest rate vector. Except for the isolated consideration of a competitive and costless banking system in Section 7.2, this centrally planed economy has only two agents, the government and a consolidated household-producer. This implies that the budget constraint of one agent, together with the equilibrium in the market for the consumption good, leads to the budget constraint of the other agent. Therefore, making $y = c$ in the household's budget constraint (1), we get the government's constraint:

$$\dot{X} + \dot{B} = iB + \langle \vec{i}, X \rangle + H$$

To complete the description of our model, we make the assumption that the government keeps a constant nominal growth of the most liquid asset in the economy (here, currency, X_{1c}), so that $\dot{X}_{1c}/X_{1c} = \Omega$. This implies that the rate of growth of the real value of currency, \dot{x}_1/x_1 , is given by $\Omega - \pi$. In the steady state, $\pi = \Omega$ and the real value of transfers (which can be positive or negative) is determined by the equation:

$$h = -(i - \pi) b - \langle i_R, x \rangle$$

Note that, except in the case of x_1 , for which $i_{R1} = -\pi$, the vector of real interest rates i_R above can assume positive or negative values, depending if the nominal interest rates of the respective monetary asset has been fixed below or above the rate of inflation.

3 Divisia Indexes

While conventional (simple-sum) monetary aggregates are not useful for welfare measurements, Divisia aggregates, which provide an adequate measure of transactions services, can perform such a function. As argued by Bruce (1977), there is a general equivalence between Divisia quantity indexes and consumer's surplus measures of welfare losses. A particular version of this general principle, as we shall demonstrate, associates Divisia indexes of monetary services with welfare costs of interest rate wedges.

Nominal Divisia indexes weigh the variations of the quantities of each monetary aggregate by its relative opportunity costs. In equilibrium, these opportunity costs are equivalent to prices, and the result is a multidimensional consumer's surplus measure. In economies where currency and differ-

ent bank deposits perform monetary services, components with high opportunity cost, which are the ones mostly frequently used for purposes of transaction (currency being a superior limiting case), are given a higher weight in the Divisia methodology. On the other hand, components with low opportunity costs (those that pay an interest rate close to the benchmark interest rate), which are the ones more likely to be held for saving services, rather than for transactions, are given a reduced weight. In this way, Divisia aggregates adequately capture the transacting motive for holding money, which, in turn, can be associated with welfare measures.

Divisia indexes also adequately take into consideration the demand shifts among different types of monetary assets. This makes them appropriate for our present purposes. Let us consider a consumer who exchanges the same dollar amount from a money-market account into currency. It is reasonable to assume, for instance, due to a maximum number of checks that can be withdrawn in a certain period of time, from the money market account, that after this portfolio reallocation the consumer will obtain a higher amount of transacting services than he did before. Indeed, the simple fact that both assets use to be held by individuals, despite having different opportunity costs, show that currency and money market accounts are not perfect substitutes in providing monetary services. Therefore, after this portfolio reallocation the consumer will be able to spend less time shopping and to dedicate more time to the production of the consumption good, which may increase welfare.

Simple-sum aggregates, because they equally weigh all components, do not change as a result of such demand shifts. Hence, simple-sum aggregates are not appropriate for assessing this type of welfare variation. Divisia indexes, on the other hand, would consistently rise with such a portfolio reallocation, since the same dollar amount would be subject to one weight before the exchange, and to another weight, higher than the first one, after the exchange. This reasoning suggests that Divisia indexes of monetary services should be negatively correlated with welfare measures. We shall see that this is indeed true.

Divisia indexes also perform better than narrow aggregates in measuring variations in transactional services and in welfare. For instance, in the case of a portfolio reallocation like the one we just mentioned, from a near-money into currency, narrow definitions of money could fail in deducting the fall of monetary services derived from the lower holdings of the money market account after the change. Indeed, because the narrow aggregate could fail to consider such near-money, only the higher holdings of currency would be taken into consideration.

Divisia indexes have been proposed by Barnett (1980) and Donovan (1978) as the adequate way to build monetary aggregates. The main characteris-

tic of Divisia monetary indexes is that the weights that are attached to the variations of the monetary balances depend upon its opportunity cost. Assuming that portfolios are in equilibrium, these weights provide a measure of the marginal liquidity services provided by the respective monetary balance.

Formally speaking, the Divisia index is a mapping from a set of paths into the real line. Different versions of the Divisia index can be found in the literature, depending upon how the nominal prices used in their construction are normalized (or deflated). In the three next Sub-sections we consider continuously differentiable paths $\chi : [a, b] \rightarrow L$ followed by the vector of monetary aggregates x , and define three different versions of Divisia indexes.

3.1 The Divisia Index Normalized by the Seigniorage (DS)

Consider the vector field defined on L :

$$F_S(u(x)) = \left(\frac{u_1}{\langle u, x \rangle}, \frac{u_2}{\langle u, x \rangle}, \dots, \frac{u_n}{\langle u, x \rangle} \right)$$

We define by DS (Divisia Seigniorage) the Divisia index when the normalization is given by the inner product $\langle u, x \rangle$:

$$DS(\chi) = \exp \int_{\chi} \langle F_S(u(x)), dx \rangle \quad (8)$$

This version of the Divisia index is the one mostly used in the literature (see, for instance, Hulten (1973) or Anderson et Al. (1997)). There are two reasons for this. First, as we shall see, this Divisia index has the nice property of exactly tracking the monetary aggregator at the optimum, a property that automatically assures its path independence. Second, it satisfies the factor reversion property that the product of the quantity and price indexes equals the total expenditure on the assets included in the index.

3.2 The Divisia Index Normalized by the Extended GDP (DE)

Alternatively, we make

$$F_E(u(x)) = \left(\frac{u_1}{1 + \langle u, x \rangle}, \frac{u_2}{1 + \langle u, x \rangle}, \dots, \frac{u_n}{1 + \langle u, x \rangle} \right)$$

and denote by DE (Divisia Extended) another version of the Divisia index, in which the weights are normalized by the extended GDP, as previously defined:

$$DE(\chi) = \int_{\chi} \langle F_E(u(x)), dx \rangle \quad (9)$$

This version of the Divisia index is found in Simonsen and Cysne (1994, 1999).

3.3 The Divisia Index Normalized by the GDP (DG)

A third version of the Divisia index, which we call DG (Divisia GDP) is presented in Bruce (1977). We define F_G by

$$F_G(u(x)) = (u_1, u_2, \dots, u_n)$$

and make:

$$DG(\chi) = \int_{\chi} \langle F_G(u(x)), dx \rangle \quad (10)$$

DG can be interpreted as a generalization of the area under a demand curve, although it is a different object from the mathematical point of view.

4 Path Independence, Financial Innovations and Price Rankings

As line integrals, Divisia indexes can suffer from the serious defect of depending on the path over which integration is taken. We shall see in this Section that all the three versions of the Divisia index here presented are path independent. The following theorems will be used in the demonstrations:

Theorem 1 (*Potential Function Theorem [PFT]*): Let $F = (k_1(x), k_2(x), \dots, k_n(x))$ be a C^1 vector field defined in an open connected set L , which is also simply connected. Then F admits a potential function λ (meaning that $F = d\lambda$) if and only if:

$$\frac{\partial k_i}{\partial x_j} = \frac{\partial k_j}{\partial x_i}, \quad i = 1, 2, \dots, n; \quad j = 1, 2, \dots, n; \quad i \neq j.$$

Proof. See Apostol (1957) or any other advanced textbook on Calculus.

■

Theorem 2 (Hulten, 1973): *Assuming that all paths are sectionally smooth, a set of necessary and sufficient conditions for path independence of the Divisia index DS is:*

- i) There exists a continuously differentiable production function $G(x)$ defined everywhere on L ;*
- ii) G is linear homogeneous in the x ;*
- iii) The level manifold of G has a price normal unique up to a scalar multiplication.*

Proof. See Hulten (1973). ■

4.1 The Divisia Index DS

4.1.1 DS and the Monetary Aggregator $G(x)$

Remark 3 below is a well known application of Hulten's theorem (see for instance Anderson et Al. (1997)). It shows that the path independence of DS in this economy is implied by the assumption of blockwise weak separability of the transacting technology, as we assumed in (3). Remark 3 establishes that the DS version of the Divisia index exactly tracks the monetary aggregator G evaluated at the optimum. The change in the aggregator function can be estimated by weighting the percentage changes in the monetary series.

Remark 3 G is equal to the Divisia index DS , up to a scalar multiplication, and DS is path independent.

Proof. $\log G(x)$ is the potential function of the vector field given by $F_S(u(x))$ in (8). In fact, given (6) and the linear homogeneity of G , one can write:

$$\langle F_S(u(x)), dx \rangle \equiv \langle \frac{u}{\langle u, x \rangle}, dx \rangle = \langle \frac{grad G}{G}, dx \rangle = \langle grad \log G, dx \rangle. \quad (11)$$

Given a , by making the normalization $G(u(x(a))) = 1$, one gets:

$$DS(\chi) = \exp \{ \log G(x(b)) - \log G(x(a)) \} = G(u(x(b)))$$

■

This result shows that the Divisia index DS is consistent with any unknown aggregator function G implied by the data.

4.1.2 Financial Innovations

The above development has been made under the assumption that the monetary aggregator is unchanging. Here we consider the case of a non-neutral technological process and analyze how DS should be adjusted to take this fact into consideration. A non-neutral progress allows us to encompass possible M1-saving innovations occurred in the transacting technology since the 80s. The analysis can be easily accomplished by assuming a transacting technology given by

$$G(\tilde{x}) = G(\delta_1 x_1, \delta_2 x_2, \dots, \delta_n x_n)$$

where $\tilde{x}_i = \delta_i x_i$, $i = 1, 2, \dots, n$, with the variables δ_i allowed to vary in order to translate productivity variations. Given the first order conditions (6), and since $\partial G / \partial x_i = \delta_i \partial G / \partial \tilde{x}_i$, it is straightforward that:

$$dG = \langle grad G, d\tilde{x} \rangle = \sum_{i=1}^n \frac{G\phi' u_i \tilde{x}_i}{\phi \delta_i} d\tilde{x}_i / \tilde{x}_i = \sum_{i=1}^n \frac{G\phi' x_i u_i}{\phi} d\tilde{x}_i / \tilde{x}_i \quad (12)$$

From the first order homogeneity of G it follows that:

$$\langle grad G, \tilde{x} \rangle = G = \sum_{i=1}^n G\phi' u_i \tilde{x}_i / \phi \delta_i = \sum_{i=1}^n G\phi' u_i x_i / \phi$$

Hence, $\phi' / \phi = (\langle u, x \rangle)^{-1}$ is an integrating factor for (12), and:

$$dG/G = \sum_{i=1}^n \frac{u_i x_i}{\langle u, x \rangle} \left(\frac{dx_i}{x_i} + \frac{d\delta_i}{\delta_i} \right)$$

Therefore, with DS_{fi} standing for DS corrected for financial innovations:

$$DS_{fi}(\chi) = \int_{\chi} dG/G = \int_{\chi} \sum_{i=1}^n \frac{u_i x_i}{\langle u, x \rangle} \left(\frac{dx_i}{x_i} + \frac{d\delta_i}{\delta_i} \right) \quad (13a)$$

The basic conclusion is that the same Divisia weights as in (11), $(u_i x_i / \langle u, x \rangle)^1$, can be used in order to track the relative variations of the monetary aggregator G . However, a second term, $(d\delta_i / \delta_i)$, which depends on the rate of growth of the productivity of each asset, must be previously added to the weighed sum.²

Although the above analytical development has been concentrated on DS , as a first approximation a similar adjustment -of keeping the weighs and including the term $d\delta_i / \delta_i$ - could be made with DE (and DG) to account for financial innovations.

¹Write $\frac{u_i}{\langle u, x \rangle} dx_i$ in (11) as $\frac{u_i x_i}{\langle u, x \rangle} dx_i / x_i$.

²Spencer (1998) arrives at this same conclusion using a static cost-minimization argument.

4.1.3 Ranking the Vector of Opportunity Costs

In the model presented in Section 2, s denotes the percentage reduction in production and consumption when the economy is not completely satiated with (socially costless) monetary services, and represents a direct measure of the welfare costs of interest rate wedges, as a fraction of GDP.

Equation (4) indicates a negative correlation between the welfare costs of interest rate wedges, s , and the aggregator function G . Therefore, Remark 3 establishes a negative correlation between s and the Divisia index DS . For instance, $\phi(s) = s$ implies $s = 1/(1 + G) = 1/(1 + DS)$ ³.

Put together, these two facts, the interpretation of s as a welfare measure, and the exactness of DS in tracking G at the optimum, can be used for price ranking purposes.

When there is only one interest rate, and therefore only one opportunity cost to be considered, the Friedman rule, when valid, states that the social optimum is achieved by making the interest rate (and the opportunity cost of currency) equal to zero. This is the case, for instance, in the shopping-time and in the currency-in-the-utility-function monetary models studied by Lucas (2000, Sections 3 and 5). The same type of rule applies, multidimensionally, in our model. If all opportunity costs tend toward zero, G tends toward infinity and s tends towards zero.

However, in economies where several opportunity costs are considered, it is not always clear which situation leads to a higher or lower welfare, since some costs can increase, and others decrease. In this case, Remark 3 can be used as a device for reducing the comparison between two different opportunity costs vector to a single scalar. The interest rate vector which leads to the highest Divisia index $DS = G$ will lead to a highest social welfare. This is a direct consequence of the costlessness of providing monetary services in this economy. The two other versions of the Divisia index here presented (DG and DE) can do as well in ranking different price vectors.

4.2 Path Independence of DG and DE

It still remains proving the path independence of the two other Divisia indexes presented. Cysne (2000) shows that separability of the transacting technology, as we are assuming here, is a necessary and sufficient condition for the independence of the DG version of the Divisia index. Cysne's result

³When $\phi(s) = s$, (7) implies $s = \langle u, x \rangle$, a result that generalizes, to an economy with n different near-monies, Lucas' (1993 and 2000) finding, of a welfare cost equal to the seigniorage.

is obtained for $n = 2$, but the extension to a higher number of assets can be made at no cost. Using the *PFT*, the necessary and sufficient condition for the path independence of DG is given by the symmetry of the matrix of cross derivatives of u :

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}, \quad i = 1, 2, \dots, n; \quad j = 1, 2, \dots, n; \quad i \neq j \quad (14)$$

From (6), $u_i = G_{x_i}(x)\phi(s)/G(x)\phi'(s)$ and $u_j = G_{x_j}(x)\phi(s)/G(x)\phi'(s)$. By taking the cross derivatives, it follows that (14) is always satisfied, since the assumed twice-differentiability of $G(x)$ implies $G_{x_i x_j} = G_{x_j x_i}$. Hence, DG is path independent.

On the other hand, notice that the path independence of DG and DS , together with the *PFT*, guaranties the path independence of DE . In fact, using the *PFT*, path independence of DE is equivalent to having, for all $i \neq j$:

$$\frac{\partial(u_i/(1+R))}{\partial x_j} = \frac{\partial(u_j/(1+R))}{\partial x_i}$$

or, equivalently, to having

$$\frac{\partial u_i}{\partial x_j}(1+R) - u_i \frac{\partial R}{\partial x_j} = \frac{\partial u_j}{\partial x_i}(1+R) - u_j \frac{\partial R}{\partial x_i} \quad (15)$$

where $R = \langle u, x \rangle$. We know from the path independence of DG that $\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}$. On the other hand, the application of this result and the *PFT* to DS implies $u_i \frac{\partial R}{\partial x_j} = u_j \frac{\partial R}{\partial x_i}$. Hence, (15) is satisfied and DE is path independent.

5 Divisia Indexes and Welfare Measures

5.1 DG and the Partial-Equilibrium Welfare Measure

Notice that

$$DG = \int_{\chi} \langle u, dx \rangle$$

corresponds to a multidimensional consumer's surplus measure and, therefore, can be interpreted as a partial-equilibrium measure of the welfare costs

of interest rate wedges. Some examples of works in the literature that use such a welfare measure, referring to the particular case of the welfare costs of inflation, are given by the pioneering contributions of Marty and Chaloupka (1988) and Marty (1994, 1999). Baltensperger and Jordan (1997) also used such a measure, quoting Marty's previous investigations on this issue. These works are concerned with other questions, and do not concentrate on the integrability problem, or on the relationships between welfare measures and Divisia indexes.

5.2 The Relation Between the Shopping Time s and the Divisia Index DE

As we noted previously, in our model s denotes the percentage reduction in production and consumption and represents a direct welfare measure, as a fraction of GDP. For empirical purposes, however, the function $\phi(s)$ is not known and, therefore, one cannot directly calculate s . This problem can be solved by eliminating $\phi(s)$ and $\phi'(s)$ from the equilibrium equations. First total differentiate (4) and use the first order conditions (6) to get:

$$-d s = G(x) \phi'(s) (\langle u, dx \rangle + d s)$$

Using (7) and (4) to eliminate $\phi(s)$ and $\phi'(s)$,

$$ds = -\frac{(1-s) \langle u, dx \rangle}{1-s + \langle u, x \rangle} \quad (16)$$

Equation (16) is an n -dimensional version, for an economy with n types of deposits, of the expression that Lucas (2000, equation 5.8) derives in his work, in the particular case when $n = 1$ (with $x_1 = x_{1c} + x_{1d} = m_1$). Lucas solves the case $n = 1$ numerically for a semi-log and for a log-log curve. Cysne (2000b) presents a closed-form solution for $n = 1$ when the money demand is log-log. When $n > 1$, (16) represents a system of n simultaneous non-separable and non-linear partial differential equations.

$$\frac{\partial s(x)}{\partial x_i} = V_i(s(x), x) \quad i = 1, 2, \dots, n$$

where

$$V_i(s(x), x) = -\frac{(1-s(x)) u_i(x)}{1-s(x) + \langle u(x), x \rangle} \quad (17)$$

A solution to this system is a function $s(x)$ that satisfies these equations identically in x . For $n > 1$, one should be aware that, when the functions $u_i(x)$ are arbitrarily-assigned, this total differential equation does not necessarily correspond to a primitive $\xi(s, x) = c$. The problem therefore arises to find conditions for this total differential to be integrable. A necessary condition comes from the symmetry of cross partial-derivatives of $s(x)$ for $i = 1, 2, \dots, n$; $j = 1, 2, \dots, n$; $i \neq j$:

$$s_{x_i x_j} = \frac{\partial V_i}{\partial s} \frac{\partial s}{\partial x_j} + \frac{\partial V_i}{\partial x_j} = s_{x_j x_i} = \frac{\partial V_j}{\partial s} \frac{\partial s}{\partial x_i} + \frac{\partial V_j}{\partial x_i} \quad (18)$$

Example 5 presents a case where the demand functions $V(x)$ plugged in (16) satisfy these integrability conditions, thereby leading to a closed form solution of s as a function of x .

As one observes from (18), when V can be made not to depend upon s , the integrability conditions turn into the much simpler form:

$$s_{x_i x_j} = \frac{\partial V_i}{\partial x_j} = s_{x_j x_i} = \frac{\partial V_j}{\partial x_i} \quad (19)$$

In our case, as one observes from (17), V does depend of s , therefore characterizing what we call a non-separable equation. However, we shall see below that $-DE$, given by (9), can be used as a reasonable approximation to s . Using $-DE$, instead of s presents the following nice features: i) one can work with the simpler integrability conditions (19), instead of (18), thereby amplifying the class of demand functions that can be used in order to recover welfare measures; ii) when the demands for the monetary assets are known, the attainment of closed-form solutions for the welfare measures is algebraically simplified; and iii) alternatively, when the demands for monetary assets are not known by the researcher, using $-DE$, which is an index, has the advantage of allowing for direct welfare calculations from market data.

Simonsen and Cysne (1994, 1999) show, for $n = 2$ and a fixed opportunity cost of the interest-bearing deposit, that (16) can be approximated by a set of simpler, separable differential equations. We follow the same type of approach here, for an economy with n monetary assets. The demonstration is somewhat similar, except for the fact that we will work with line integrals defined in the n -dimensional space, instead of ordinary integrals in the one-dimensional space.

Proposition 4 *We consider paths $\chi^{**} : t \rightarrow x(t)$, $t \in [a, b]$, $\chi^{**}[a, b] \subset L$, with $\langle u, dx \rangle < 0$. A particular case occurs for paths χ^* with $\lim_{t \rightarrow a} x_i = +\infty$*

and $x'_i(t) < 0$ for $i = 1, 2, \dots, n$. Letting $s(x)$ denote the solution to (16) along such a path,

(1)

$$1 - e^{DE(x)} < s(x) < -DE(x) \quad (20)$$

(2) For low values of DE ;

$$\frac{-DE(x) - (1 - \exp^{DE(x)})}{s(x)} \approx \frac{-DE(x)^2}{2s(x)} \approx \frac{-DE(x)}{2} \quad (21)$$

Proof. See Appendix ■

Note that our results from Section 4 guarantee that DE satisfies the integrability conditions given by (19).

5.3 The Relation Between DE and DG

The difference between DG and DE is due to the existence of the term $1 + \langle u, x \rangle$ in the denominator of DE . The reason for this distinction is that DE , having been derived as an approximation to ds , takes into consideration the income variations caused by the interest rate wedges, whereas the partial-equilibrium measure does not. Indeed, taking the total derivative of (2), and using the first order conditions (6), leads to $dc = N_s(\langle u_i, dx_i \rangle + ds)$. If income is compensated and $dy = dc = 0$, we get the total differential $ds|_{dc=0} = -\langle u_i, dx_i|_{dc=0} \rangle = -dDG|_{dc=0}$. Since $ds < -dDE < -dDG$, this implies that $dc = 0$ makes $DE = DG$ ⁽⁴⁾. The same observation (income effect) explains the distinction between Lucas' expression 5.8 (equation (16) with $n=1$) and Bailey's expression $dB = -u_1 dx_1$ for the welfare costs of inflation in a currency-only economy.

When $\langle u, x \rangle \rightarrow 0$, $-DG \searrow -DE$. $-DG$ is an upper bound to $-DE$, which in turn is an upper bound to s . Inequalities (20) can then be written:

$$1 - e^{DE} < s < -DE < -DG \quad (22)$$

This result that has been initially derived by Simonsen and Cysne in the special case of only two monetary assets, and where the opportunity cost of deposits was supposed to be constant.

⁴I owe this observation to a related remark made by Samuel Pessoa.

6 Measuring Welfare Costs in Empirical Research

Suppose that a researcher is interested in measuring welfare costs or gains associated with a change of the vector of opportunity costs from $u(x(a))$ to $u(x(b))$. The researcher does not know the transacting technology, but, from empirical observation, has estimated the assets demand functions. The evaluation of the welfare variation can then be assessed by (16), (9) or (10), as we show in the following example.

Example 5 *We assume that $\phi(s) = s$ and that the transacting technology (which is not known by the researcher) is given by:*

$$\begin{aligned} c &= G(x)s = Ax_1^{a_1}x_2^{a_2}\dots x_n^{a_n}s \\ 1 &= a_1 + a_2 + \dots + a_n, \quad A > 0 \end{aligned}$$

Suppose, in addition, that the researcher has been able to properly estimate the demand functions compatible with this technology (the case when the demands are not known is explored in the next subsection):

$$u_i = \frac{a_i/x_i}{1+G}, \quad i = 1, 2, \dots, n \quad (23)$$

Our objective is recovering the measure s (or some good approximation of it) departing from the knowledge of these demands. As before, we consider the path $\chi^ \subset L$. The first option is plugging equations (23) directly into (16) and solving the system of non-separable partial differential equations given by:*

$$ds = -\frac{(1-s)\sum_{i=1}^n \frac{a_i dx_i}{x_i(1+G)}}{1-s+1/(1+G)} \quad (24)$$

which in this case leads to the closed-form solution:

$$s(x) = \frac{1}{1+G(x)} \quad (25)$$

The problem with this alternative is that, in general, providing a closed-form solution to the non-separable partial differential equation (16) can be a non-trivial or even impossible task, depending on the assets demand functions that are plugged into (16). Alternatively, we suggest using the approximation $-DE$ given by (9), a procedure which is allowed by inequalities (21) and (22). By observing (18) and (18) one notices that the integrability conditions

are much less demanding when one uses (9) than when one uses (16). This makes the integration process easier and, besides, enlarge the class of demand functions that can be used by the researcher in order to get the desired welfare measure (since now we are allowing for the use not only of s , but also of $-DE$). In fact, there may be demand systems which satisfy (19), but not (18).

In this specific example, using (23) in (9):

$$DE(x) = -\frac{1}{2} \log(1 + \frac{2}{G}) \quad (26)$$

For the purpose of comparison, we also present the expression for DG :

$$DG(x) = -\log(1 + \frac{1}{G}) \quad (27)$$

(25), (26) and (27) satisfy the inequalities (22). In fact,

$$1 - e^{DE} < s \Leftrightarrow \frac{G}{1+G} < (\frac{G}{G+2})^{1/2} \quad (28)$$

$$s < -DE \Leftrightarrow \log(1 + \frac{2}{G}) > \frac{2}{(1+G)} \quad (29)$$

$$-DE < -DG \Leftrightarrow \frac{G}{1+G} < (\frac{G}{G+2})^{1/2} \quad (30)$$

Inequalities (28) and (30) are trivially satisfied, since we know that $G > 0$. To obtain inequality (29), make $E = 1/G$ and note that:

$$\log(1 + \frac{2}{G}) > \frac{2}{(1+G)} \Leftrightarrow \log(1 + 2E) > \frac{2E}{(1+E)}$$

By making $f(E) = \log(1 + 2E) - \frac{2E}{(1+E)}$, one observes that $f(0) = 0$ and that $f'(E) = E^2/(1+E)^2(1+2E) > 0$ for any $E > 0$. It follows that, for any $E > 0$, $f(E) > 0$.

In order to illustrate these results with a numerical example, consider a situation where $n = 3$, $A = 2000$, $a_1 = 0.5$, $a_2 = 0.3$, $a_3 = 0.2$. We assume

the initial values of the monetary aggregates to be given by $x_1(1) = 0.090$, $x_2(1) = 0.058$, $x_3(1) = 0.045$, and the final values by $x_1(2) = 0.053$, $x_2(2) = 0.032$, and $x_3(2) = 0.022$. The implied values of the opportunity costs in the first and second situations are given by, respectively, $u(1) = (4.0156\%, 3.7387\%, 3.2125\%)$ and $u(2) = (12.1857\%, 12.1095\%, 11.7426\%)$. In this case

we get, as a percentage of GDP:

	$1 - e^{DE}$	s	$-DE$	$-DG$
<i>Initial</i>	0.7202	0.7228	0.7228	0.7254
<i>Final</i>	1.2834	1.2917	1.2918	1.3001
<i>Variation</i>	0.5632	0.5689	0.5689	0.5747

If the researcher chooses to solve the non-separable partial differential equation, he will find a welfare cost figure, due to the change of the vector of opportunity costs, from $u(1)$ to $u(2)$, of 0.56886% of GDP. Alternatively, the use of the Divisia index $-DE$, leads to the figure of 0.56892% of GDP, a negligible difference.

6.1 The Case of Unknown Demands

In the example above, we assumed the exact functional specification and the parameters of the assets demand functions to be known by the researcher. When this is possible, one only needs to rely on quantity data and on the estimated parameters of the underlying demand functions.

When this is not possible, or is too costly, the real impact of the results derived here is in terms of the discrete counterpart of (9). Indeed, one nice feature of using the Divisia index $-DE$ as a welfare measure is that it can always be computed, given observations on the interest rates and the monetary aggregates. Statistical index numbers do not depend on any unknown parameters. The use of market prices compensate for the absence of knowledge about parameters or functional specifications. Prices (here, opportunity costs) and quantities have the advantage of being directly observable.

Since collecting data in continuous time is impossible, we have to rely on some approximation of (9) defined in discrete time. DED , below, provides one such possible approximation:

$$DED(t) - DED(t-1) = - \sum_{j=1}^n w_{t,j}^* [x_{j,t} - x_{j,t-1}] \quad (31)$$

where

$$w_t^* = \frac{1}{2} \left(\frac{u_1}{1 + \langle u, x \rangle}(t) + \frac{u_1}{1 + \langle u, x \rangle}(t-1) \right)$$

DED consistently approaches DE as Δt goes to zero. If we use this formula to make a rough approximation of DE , based only on the initial and final values of the variables, we get, for the parameter values of our previous

example, $\Delta DE = 0.6728$, as against the value $\Delta DE = 0.5689$ previously calculated. Of course, the approximation could be improved by the use of additional quantity and price data observations between the two periods of reference. Additional improvements can also be obtained by the use of numerical-methods techniques that, relatively to (31), better approximate DE in discrete time.

7 Welfare Costs of Inflation

The connection between the welfare costs of inflation and Divisia indexes of monetary services has been advanced by Lucas (2000, p. 270), who wrote: “I share the widely held opinion that M1 is too narrow an aggregate for this period [the 1990s], and I think that the Divisia approach offers much the best prospects for resolving this difficulty”. However, Lucas (2000) does not explicitly present the link between the partial- and general-equilibrium measures of the costs of inflation, derived in his work, and the Divisia index of monetary services.

Also referring to the calculation of the welfare costs of inflation, Marty (1999, p.46) notices that “if M1 is used as a relevant money supply, some correction must be made for the interest paid on portions of M1”.

In this Section we make some hypotheses that connect interest-rate wedges with the rate of inflation, thereby allowing investigations of questions specifically related to the welfare costs of inflation to be addressed with the results here derived.

It follows from (5) that our economy is a Fisherian one, where the benchmark interest rate is determined by the rate of inflation, which is endogenous in the model, and by the rate of time preference. Under these circumstances, interest rate wedges can be assumed to be determined by the rate of inflation in at least three alternative cases. In the first two of them, it does not matter, for our purposes here, if the monetary assets are directly issued by the government (as we assumed when modeling the government in Section 2), or by a regulated costless banking system. These are: i) when all monetary assets pay no interests, such as in currency-only economies or in economies with currency and non-interest-bearing deposits only; ii) when the interest rate paid by the different near-moneys is supposed to be exogenously determined by the government (a generalization of (i)). The third case assumes the monetary assets to be issued by a competitive and costless banking system. In the following two subsections we investigate these alternatives.

7.1 Fixed Interest Rates

With interests on deposits fixed by the government, directly or through a regulated banking system, (case (i) ,which is more usual, or case (ii)), the gap between the deposits rate and the bond rate increases pari-passu with the rate of inflation. The following example illustrates how our results can be used in this case:

Example 6 *We assume the same particular transacting technology, the same demand functions and the same values of the parameters n , A , a_1 , a_2 , and a_3*

of the previous example. We make the annual inflation rate (π) vary from 0.0% to 2.0 (200%). With the rate of time preference $\rho = 0.02$, the nominal interest rate of the benchmark asset varies from 0.02 to 2.02. Since currency (x_1 in this example), by definition, pays a nominal interest rate equal to zero, its opportunity cost (u_1) will also vary in the same range. The two other assets, x_2 and x_3 , are assumed to pay annual fixed nominal interest rates equal to, respectively, 0.003 (0.3%) and 0.008, (0.8%), in which case their opportunity costs will vary from 0.017 to 2.017 (x_2), and from 0.012 to 2.012 (x_3), respectively.

The table below presents the values of the different measures of the welfare costs of inflation as a percentage of GDP. We add one additional point to the series, assuming the economy to be satiated with monetary services for an annual rate of deflation equal to 2%.

<i>Inflation</i>	$1 - e^{DE}$	s	$-DE$	$-DG$
-2%	0.0000	0.0000	0.0000	0.0000
0%	0.4883	0.4895	0.4895	0.4907
5%	0.9623	0.9669	0.9669	0.9716
10%	1.2662	1.2742	1.2743	1.2824
20%	1.7151	1.7298	1.7300	1.7449
50%	2.6213	2.6557	2.6563	2.6916
100%	3.6376	3.7038	3.7054	3.7741
150%	4.4073	4.5043	4.5073	4.6089
200%	5.0481	5.1753	5.1800	5.3141

One can observe that the difference between s and $-DE$ is immaterial, even for values of $-DE$ not so close to zero. Figures 1 and 2⁵ present the evolution of the different welfare measures and their differences, relatively to s , for annual rates of inflation ranging from 0.0% to 200%.

⁵In Figure 1, we call $1 - e^{DE}$ Lower Bound and $-DG$ Upper Bound. In Figure 2, NDG stands for $-DG$ and LB for lower bound ($1 - e^{DE}$).

Besides the convergence of $i - e^{DE}$, s , $-DE$ and $-DG$ to zero, for low rates of inflation, it becomes clear that, as inflation rises, both the difference between s and $1 - \exp(DE)$, and between $-DG$ and s , increase. The same happens to $-DE - s$, but at a significantly lower rate.

7.2 The Competitive Case

In this Subsection, only, we assume that the monetary assets are issued by a costless and competitive banking system, and not by the government, as assumed till now. Our only purpose is showing that, as anticipated by Bailey (1956, p. 104), for the case that he describes as “banks operating rationally”, only the monetary base and the benchmark interest rate needs to be considered for the purpose of measuring the welfare costs of inflation. The reason is that, in this case, given the vector of reserve requirement ratios \vec{k} , the interest rate-wedge vector (or price vector) ($u(t = 1) = \vec{k}i_1$) is always proportional to some fixed base price vector ($u(t = 0) = \vec{k}i_0$), the so called Hicksian separability.

The interest rate wedge of each bank deposit exactly equals the reserve requirement ratio times the benchmark rate of interest. Therefore, the multiplication of the interest rate wedge u_i by the infinitesimal variation of the respective monetary aggregate, (the term $u_j dx_j$ in (9)), exactly equals the benchmark interest rate (i) times the infinitesimal variation of the fraction of the asset that is maintained as a monetary liability of the Central Bank, as reserve requirement ($u_j dx_j = (k_j i) dx_j = i(k_j dx)$).

Since, in this formulation, currency actually behaves like a deposit with a hundred percent reserve ratio (remember that

$u_{\text{currency}} = 1 \cdot i = i$), one concludes that the welfare costs of interest rate wedges can be properly measured considering only the monetary base (Z) and the benchmark interest rate i .

Formally:

$$\begin{aligned}
 -DE(u(x)) &= - \int_x \sum_{j=1}^n \frac{k_j i dx_j}{1 + \langle k i, x \rangle} \\
 &= - \int_x \sum_{j=1}^n \frac{i(k_j dx_j)}{1 + i \langle k, x \rangle} \\
 &= - \int_x \frac{i}{1 + iZ} dZ
 \end{aligned} \tag{32}$$

Therefore, as Bailey, argues, in this case (competitive and costless banking system) only the monetary base and the benchmark interest rate need to be considered for the purpose of calculating welfare costs. However, notice that there is one important quantitative asymmetry. While Bailey uses the welfare measure

$$dB = - \int_{m_0}^{m_1} i dZ \quad (33)$$

(which corresponds to our Divisia index DG when $n = 1$ — see Section 5.3—), our result (32) includes the integration factor (which corresponds to the marginal utility of income) $i/(1 + iZ)$.

Bailey argues that, in the alternative case, in which banks do not charge the economic rate of interest for their loans, or do not pay market interest for their deposits (which would correspond to the non-rational situation), m_1 , and not the monetary base, should be considered for the purpose of assessing the welfare costs of inflation. In his words, referring to this case: “the welfare costs of inflation will be the same for a given inflation regardless of what fraction of the money supply is currency”. This is also qualitatively compatible with our conclusions. In fact, if a deposit x_j pays no interest, its opportunity cost will be equal to i and, from either (16) or (9), it will be grouped together with currency in the welfare cost expression. Fractions of currency and non-interest bearing deposits in the means of payment (m_1) will not matter.

8 Concluding Remarks

We consider economies where currency and interest-bearing deposits perform monetary functions and investigate the welfare costs of interest rate wedges. Under some conditions, the results obtained can also be helpful in the investigation of the welfare costs of inflation.

We show that the extension of Lucas’ equation for the welfare costs of inflation leads to a system of simultaneous non-separable partial differential equations from which it can be algebraically hard, if not impossible, sometimes, to arrive at closed-form welfare expressions.

As a more algebraically and empirically suitable alternative, we suggest an approximation of the welfare cost expression by a Divisia line integral. We prove the path independence of this line integral and show that the maximum relative error involved in the approximation is generally negligible.

The alternative is algebraically more suitable because, when the assets demand functions are known, the integrability conditions of the Divisia line integrals are less restrictive, and the integration process more straightforward, than the one associated with non-separable partial differential equations.

On the other hand, when the assets demand curves are not known, the alternative of using a discrete index number allows the researcher to make approximate welfare measurements and to construct price rankings based only on direct market observations of interest rates and monetary aggregates.

Appendix

Proof. (Proposition 4)

(a) Along the paths χ^{**} considered, since $0 < s < 1$ and $\dot{x}(t) < \vec{0}$, we can write:

$$\begin{aligned}
 \int_{\chi^*} ds &= \int_a^b -\frac{\langle u(x(t)), \dot{x}(t) \rangle}{1 + \langle u(x(t)), x(t) \rangle / (1 - s(x(t)))} dt \\
 &< \int_a^b -\frac{\langle u(x(t)), \dot{x}(t) \rangle}{1 + \langle u(x(t)), x(t) \rangle} dt \\
 &< \int_a^b -\frac{\langle u(x(t)), \dot{x}(t) \rangle}{1 - s(x(t)) + \langle u(x(t)), x(t) \rangle} dt \\
 &= \int_{\chi^*} \frac{ds}{1 - s}
 \end{aligned}$$

(20) follows from the above inequalities by noticing that: (1) the third term in the above expressions is equal to $-DE$; (2) $\lim_{x \rightarrow \infty} s = 0$ and (3) $\int_{\chi^*} \frac{ds}{1-s} = -\ln(1-s) \big|_{\infty}^x = -\ln(1-s(x))$.

(b) The second part of the Proposition (equation (21)) is obtained by first taking a second-order Taylor approximation to the exponential function. This makes $-DE - (1 - \exp(DE)) \leq DE^2/2s$. Using L'Hôpital's Rule in (9) and (16), one concludes that DE/s tends toward one when x (or any of its components) tends toward infinity. Therefore, as the components of x increase, $DE \searrow 0$ and $DE^2/2s$ tends towards $DE/2$. ■

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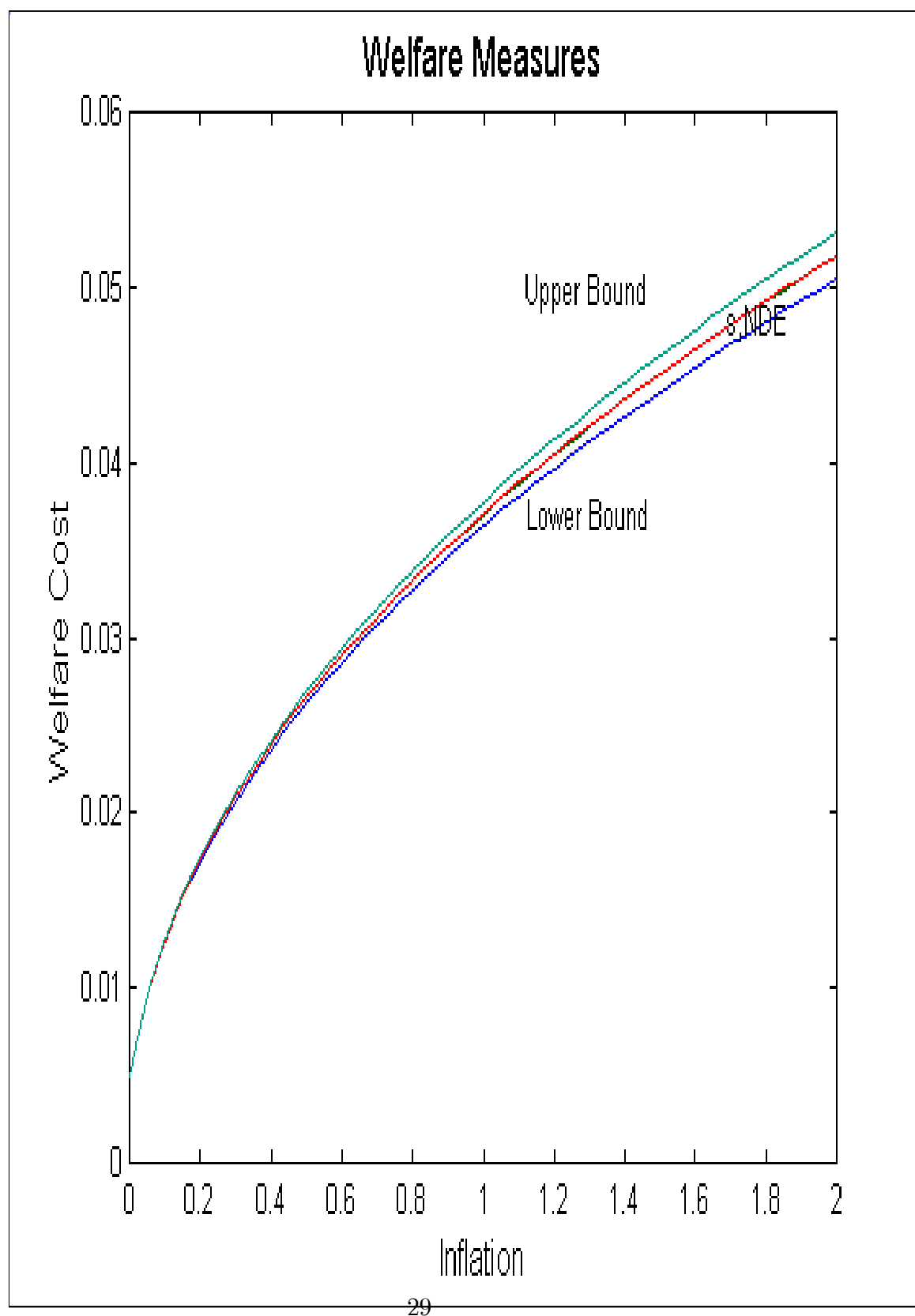


Figure 1: Figure 1

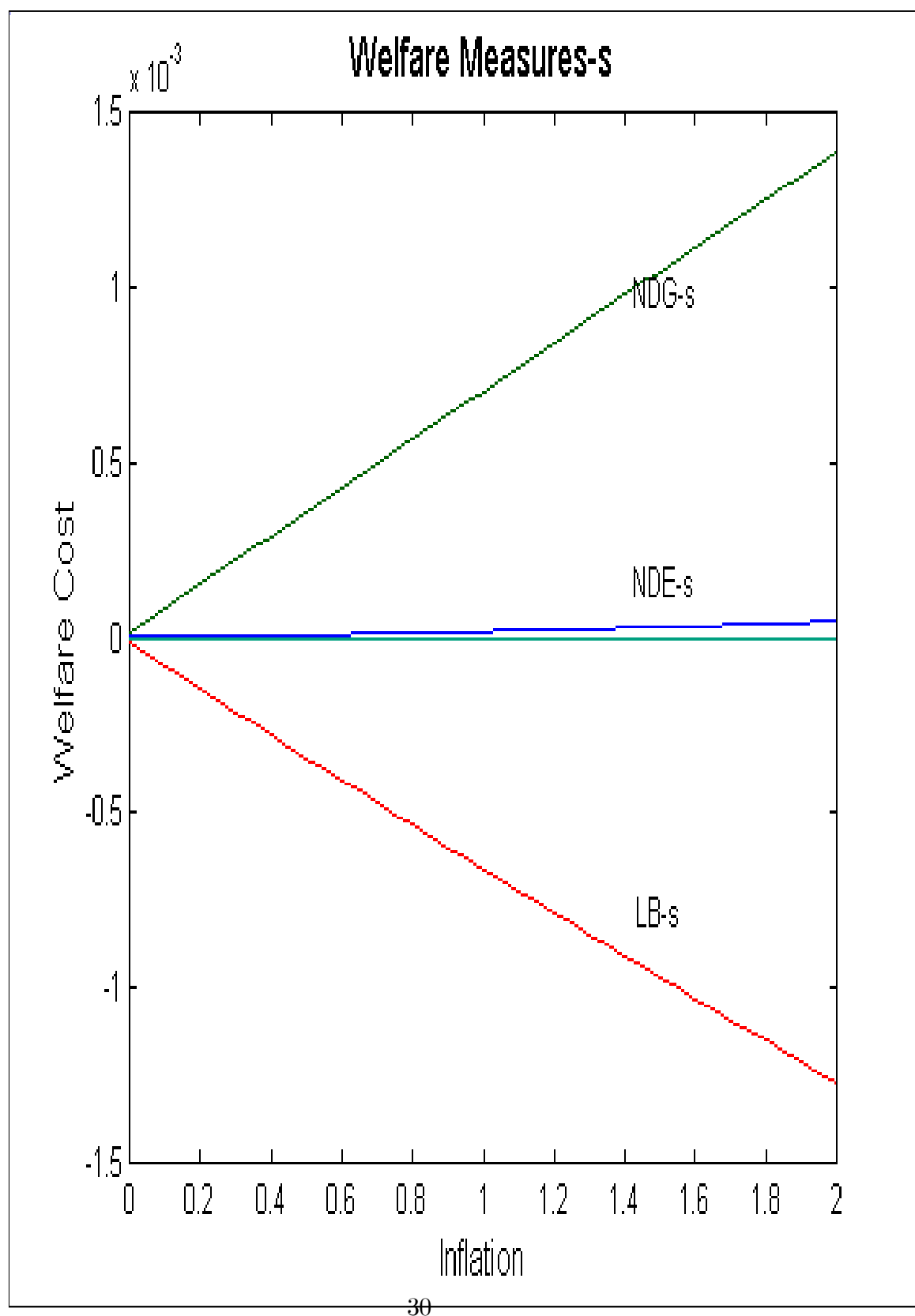


Figure 2: